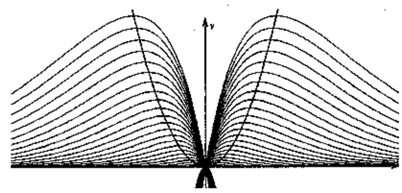
1a $\lim\limits_{x\to\infty}(a x e^{-x/a}) = \lim\limits_{x\to\infty}\dfrac{ax}{e^{x/a}} \underset{\substack{\text{Bernoulli}\\ \text{L'Hospital}}}{=} \lim\limits_{x\to\infty}\dfrac{a}{\frac{1}{a}e^{x/a}} = a^2 \lim\limits_{x\to\infty}\dfrac{1}{e^{x/a}} = 0$

1b $\int\limits_0^\infty x e^{-x}\,dx \underset{\substack{\text{partielle}\\ \text{Integration}}}{=} \Big[-xe^{-x}\Big]_0^\infty - \int\limits_0^\infty(-e^{-x})\,dx = \Big[-xe^{-x}\Big]_0^\infty - \Big[e^{-x}\Big]_0^\infty = \underset{\substack{\text{siehe}\\ \text{Aufg.1a}}}{0} - 0 - 0 + 1 = \underline{\underline{1}}$

1c $f'(x) = (a-x)\cdot e^{-x/a} = 0$

$f'(x) = (a-x)\cdot e^{-x/a} = 0 \quad \Rightarrow$

$\Rightarrow \left.\begin{array}{l} x = a \\ y = f(a) = \frac{1}{e}\cdot a^2 \end{array}\right\} \Rightarrow \underline{\underline{y = \tfrac{1}{e}\cdot x^2}}$



1d $f''(x) = (\tfrac{x}{a}-2)\cdot e^{-x/a} = 0 \quad \Rightarrow \quad x = 2a \;;\; y = f(2a) = 2a^2e^{-2}$

$y = x \quad \Leftrightarrow \quad 2a^2e^{-2} = 2a \quad \underset{(a\neq 0)}{\Rightarrow} \quad \underline{\underline{a = e^2}}$

1e $f'(x) = -(x-a)\cdot e^{-x/a} \quad ; \quad f''(x) = \tfrac{1}{a}(x-2a)\cdot e^{-x/a} \quad ; \quad f'''(x) = -\tfrac{1}{a^2}(x-3a)\cdot e^{-x/a}$

Vermutung: $f^{(n)}(x) = (-1)^n\frac{1}{a^{n-1}}(x-na)\cdot e^{-x/a}$

Die Verankerung ist für n=1, 2, 3 schon geleistet. Induktion:

$f^{(n+1)}(x) = \left(f^{(n)}(x)\right)' = \left((-1)^n\frac{1}{a^{n-1}}(x-na)\cdot e^{-x/a}\right)' =$

$= (-1)^n\frac{1}{a^{n-1}}\cdot\left(1\cdot e^{-x/a} + (x-na)\cdot e^{-x/a}\cdot(-\tfrac{1}{a})\right) = \ldots = (-1)^{n+1}\frac{1}{a^{(n+1)-1}}\left(x-(n+1)a\right)\cdot e^{-x/a}$

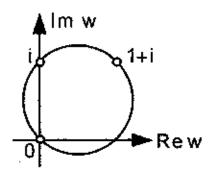
2a $f(3+4i) = \frac{(3+4i)+i}{(3+4i)+1} = \frac{3+5i}{4+4i} = \frac{(3+5i)(4-4i)}{(4+4i)(4-4i)} = \frac{32+8i}{32} = \underline{\underline{1+\tfrac{1}{4}i}}$

2b $\frac{z+i}{z+1} = w \quad \Leftrightarrow \quad \ldots \quad \Leftrightarrow \quad z = \frac{-w+i}{w-1} = \frac{-(2+3i)+i}{(2+3i)-1} = \frac{-2-2i}{1+3i} = \frac{(-2-2i)(1-3i)}{(1+3i)(1-3i)} = \frac{-8+4i}{10} = \underline{\underline{-\tfrac{4}{5}+\tfrac{4}{5}i}}$

2c $\frac{z+i}{z+1} = z \quad \Leftrightarrow \quad z+i = z(z+1) \quad \Leftrightarrow \quad z^2 = i = \text{cis}90° \quad \Rightarrow \quad z = \pm\text{cis}45° = \underline{\underline{\pm\tfrac{\sqrt{2}}{2}(1+i)}}$

2d $z = \underbrace{x}_{\in\mathbb{R}} + 0i \quad \Rightarrow \quad w = \underbrace{u}_{\in\mathbb{R}} + \underbrace{v}_{\in\mathbb{R}}i = \frac{x+i}{x+1} \quad \Rightarrow \quad \left.\begin{array}{l} u = \frac{x}{x+1} \\ v = \frac{1}{x+1} \end{array}\right\} \Rightarrow \underbrace{u+v=1}_{\text{Bildgerade}} \quad \Rightarrow \quad \underline{\underline{(1|0)\;;\;(0|1)}}.$

2e $f(-i) = \frac{-i+i}{-i+1} = 0 \quad ; \quad f(0) = \frac{0+i}{0+1} = i \quad ; \quad f(i) = \frac{i+i}{i+1} = 1+i$



3a $P_a = \frac{1}{8}\cdot\frac{7}{8} + \frac{7}{8}\cdot\frac{1}{8} = \underline{\underline{\tfrac{14}{64}}}$

3b $P_b = \binom{16}{2}\cdot(\tfrac{1}{8})^2\cdot(\tfrac{7}{8})^{14} = \underline{\underline{0.289}}$

3c Höchstens 2 defekt: $P_c = \binom{6}{0}\cdot(\tfrac{1}{8})^0\cdot(\tfrac{7}{8})^6 + \binom{6}{1}\cdot(\tfrac{1}{8})^1\cdot(\tfrac{7}{8})^5 + \binom{6}{2}\cdot(\tfrac{1}{8})^2\cdot(\tfrac{7}{8})^4 = \underline{\underline{0.971}}$

3d $1-(\tfrac{7}{8})^n \geq 0.99 \quad \Leftrightarrow \quad (\tfrac{7}{8})^n \leq 0.01 \quad \Leftrightarrow \quad n \geq \frac{\lg(0.01)}{\lg(7/8)} = 34.4\ldots$. Man muss <u>35 oder mehr</u> wählen.

3e $10'000\cdot\left(\tfrac{1}{8}\cdot 80\% + \tfrac{7}{8}\cdot 10\%\right) = \underline{\underline{1875 \text{ Stück}}}$.

4a $\cos\gamma = \dfrac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}|\cdot|\overrightarrow{AC}|} = \dfrac{\begin{pmatrix}0\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}}{\sqrt{0^2+2^2+(-1)^2}\cdot\sqrt{1^2+0^2+1^2}} = \dfrac{1}{\sqrt{10}} \Rightarrow \underline{\underline{\gamma = 71.6°}}$

4b Normalenvektor $\vec{n} = \overrightarrow{AB}\times\overrightarrow{AC} = \begin{pmatrix}0\\2\\-1\end{pmatrix}\times\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}2\\-1\\-2\end{pmatrix} \Rightarrow E:\ 2x-y-2z+D=0$.

A einsetzen liefert $\underline{\underline{E:\ 2x-y-2z-1=0}}$

4c $\vec{r} = \overrightarrow{0D} + t\cdot\vec{n} \Rightarrow \underline{\underline{\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-3\\3\\-4\end{pmatrix} + t\cdot\begin{pmatrix}2\\-1\\-2\end{pmatrix}}}$

4d Hesseform: $R = \left|\dfrac{2x_D - y_D - 2z_D - 1}{\sqrt{2^2+(-1)^2+(-2)^2}}\right| = \left|\dfrac{2\cdot(-3)-3-2\cdot(-4)-1}{\sqrt{4+1+4}}\right| = \underline{\underline{\dfrac{2}{3}}}$

4e $\left.\begin{array}{l} G = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| \underset{\text{siehe 4b}}{=} \frac{1}{2}\cdot\left|\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right| = \frac{1}{2}\sqrt{4+1+4} = \frac{3}{2} \\ h = R = \frac{2}{3} \text{ (Radius aus Aufgabe d)} \end{array}\right\} \Rightarrow V = \frac{1}{3}Gh = \underline{\underline{\frac{1}{3}}}$

oder mit Spatprodukt: $V = \frac{1}{6}\left|\left(\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}\right)\right| = \frac{1}{6}\left|\left(\underbrace{\overrightarrow{AB}\times\overrightarrow{AC}}_{\text{siehe Aufg. 4b}}\right)\cdot\overrightarrow{AD}\right| = \frac{1}{6}\left|\begin{pmatrix}2\\-1\\-2\end{pmatrix}\cdot\begin{pmatrix}-3\\2\\-3\end{pmatrix}\right| = \frac{1}{6}\cdot 2 = \underline{\underline{\frac{1}{3}}}$.

5a $\lg a = 2003\cdot\lg 2003 = 6613.26... \Rightarrow a = 10^{6613.26...} = \underbrace{10^{0.26...}}_{1.8...}\cdot 10^{6613} \Rightarrow \underline{\underline{6614 \text{ Stellen}}}$

Die Endziffer einer Potenz von 2003 entspricht der Endziffer derselben Potenz von 3.
Es genügt daher, die Endziffern von Dreierpotenzen zu beobachten:
$3^0 = 1,\ 3^1 = \underline{3},\ 3^2 = \underline{9},\ 3^3 = 2\underline{7},\ 3^4 = 8\underline{1},\ 3^5 = 24\underline{3},\ ...\underline{9},\ ...\underline{7},\ ...\underline{1},\ ...\underline{3}$ etc.
Sie durchlaufen einen Viererzyklus. Ist der Exponent durch 4 teilbar, so lautet die Endziffer 1.
$2003^{2000} = ...\underline{1},\ 2003^{2001} = ...\underline{3},\ 2003^{2002} = ...\underline{9},\ \underline{\underline{2003^{2003} = ...7}}$.

5b R hat von der Geraden $a = (OA)$ den Abstand $r = \dfrac{\left|\overrightarrow{OR}\times\overrightarrow{OA}\right|}{\left|\overrightarrow{OA}\right|} = \dfrac{55}{\sqrt{2^2+6^2+9^2}} = 5$. Das Objekt ist also die (unendlich lange) Rotationszylinderfläche mit Achse $\underline{\underline{a = (OA)}}$ und der Radius $\underline{\underline{r = 5}}$.

5c $\underline{\underline{y = \frac{1}{2}\cos(2\pi x)+1}}$

5d $4^x > 16^{(x^2-3)} \Leftrightarrow 4^x > 4^{2(x^2-3)} \underset{\substack{\text{Exponentialfunktion}\\\text{monoton wachsend}}}{\Leftrightarrow} x > 2(x^2-3) \Leftrightarrow$

$\Leftrightarrow \underbrace{-2x^2+x+3}_{\substack{\text{nach unten offene Parabel}\\\text{mit Nullstellen } -1 \text{ und } \frac{3}{2}}} > 0 \Leftrightarrow \underline{\underline{-1 < x < \frac{3}{2}}}$

5e Behauptet wird, dass im Schnittpunkt das Produkt der Steigungen -1 sei.

Überall gilt $(\cos x)'\cdot(\tan x)' = (-\sin x)\cdot\frac{1}{\cos^2 x} = -\frac{\sin x}{\cos x\cdot\cos x} = -\frac{\tan x}{\cos x}$.

Im Schnittpunkt ist $\cos x = \tan x$, folglich $(\cos x)'\cdot(\tan x)' = -1$, was zu zeigen war.