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1. $f_k(x) = \ln x^k = k \cdot \ln x$

a) $P(e^2|e) \in f_k$

$e = k \cdot \ln e^2$

$e = k \cdot 2$

$\underline{k = \frac{e}{2}}$

b) $f_k'(x) = \frac{k}{x}$

NST: $f(x) = 0$

$x = 1$

$f_k'(1) = \underline{k} = \tan 60° = \underline{\sqrt{3}}$

c) $m = f_k'(x)$

$\frac{f_k(x) - 0}{x - 0} = \frac{k}{x}$

$k \ln x = k$

$\underline{x = e}$ $\underline{y = k}$ $\underline{B(e|k)}$

d) $F(x) = kx(\ln x - 1)$

$\underline{F'(x)} = k(\ln x - 1) + kx\frac{1}{x}$

$= k \ln x - k + k$

$= k \ln x = \underline{f_k(x)}$ also ist $F(x)$ Stammfunktion zu $f(x)$

e) $\int_1^e f_k(x)\,dx = 1$

$[kx(\ln x - 1)]_1^e = 1$

$ke\underbrace{(\ln e - 1)}_{=0} - k(\ln 1 - 1) = 1$

$\underline{+k = 1}$

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2. $cis(x) = \cos x + i \sin x$

a) $2\,cis(45°) + 2\,cis(135°)$

$= 2(\cos 45° + i \sin 45°) + 2(\cos 135° + i \sin 135°)$

$= 2(\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}\,i) + 2(-\frac{1}{2}\sqrt{2} + i\frac{1}{2}\sqrt{2})$

$= 2\sqrt{2}\,i = \underline{2\sqrt{2}\,cis\,90°}$

b) $z^4 = 16\,cis\,256°$

$z_k = \sqrt[4]{16} \cdot cis\left(\frac{256° + k\cdot 360°}{4}\right) \qquad k = 0, 1, 2, 3$

$z_0 = 2\cdot cis\,64°$; $z_1 = 2\cdot cis\,154°$; $z_2 = 2\cdot cis\,244°$; $z_3 = 2\cdot cis\,334°$

c) $w = f(z) = (3+4i)z + (1+2i)$

$z = \bar{f}(w) = u\cdot w + v \qquad z = \frac{w - (1+2i)}{3+4i}$

$= \frac{1}{3+4i}w - \frac{1+2i}{3+4i}$

$= \frac{3-4i}{9+16}w - \frac{(1+2i)(3-4i)}{9+16}$

$= \underbrace{\left(\frac{3}{25} - \frac{4}{25}i\right)}_{u} w \underbrace{- \left(\frac{11}{25} + \frac{2}{25}i\right)}_{v}$

d) $z = a+bi \qquad f(a+bi) = (3+4i)(a+bi) + (1+2i) = (a+bi)$

$\underbrace{-2a + 4b - 1}_{\text{I.} \; = 0} + (\underbrace{-4a - 2b - 2}_{\text{II} \; = 0})i = 0$

I $\quad -2a + 4b = 1$

II $\quad -4a - 2b = 2$

$a = -\frac{1}{2}$; $b = 0$

$\underline{z_{Fix} = -\frac{1}{2}}$

e) k: $M(3|-2i)$; $R = 2 \rightarrow P(3|0) \in k$

$f(3-2i) = (3+4i)(3-2i) + 1 + 2i = 18 + 8i = M'$

$f(3) = (3+4i)\cdot 3 + 1 + 2i = 10 + 14i = P'$

$R' = \sqrt{8^2 + 6^2} = 10 = P'M'$

$\underline{M^*: 18 + 8i}$

$\underline{R^*: 10}$

3. $A(-4|0|0)$; $B(0|-4|0)$; $C(4|0|0)$; $D(0|4|0)$; $E(0|0|6)$

a) $\alpha[A,B,E]$: $\vec{X} = \begin{pmatrix}-4\\0\\0\end{pmatrix} + \lambda \underbrace{\begin{pmatrix}4\\-4\\0\end{pmatrix}}_{\vec{AB}} + \mu \underbrace{\begin{pmatrix}4\\0\\6\end{pmatrix}}_{\vec{AE}}$

$\beta[C,D,E]$: $\vec{CD} = \begin{pmatrix}-4\\4\\0\end{pmatrix} \sim \begin{pmatrix}-1\\1\\0\end{pmatrix}$, $\vec{CE} = \begin{pmatrix}-4\\0\\6\end{pmatrix} \sim \begin{pmatrix}-2\\0\\3\end{pmatrix}$ $\Big\}$ $\begin{pmatrix}-1\\1\\0\end{pmatrix} \times \begin{pmatrix}-2\\0\\3\end{pmatrix} = \begin{pmatrix}3\\3\\2\end{pmatrix} = \vec{n}_\beta$

$3x + 3y + 2z + d = 0$

$(-4|0|0)$: $-12 + d = 0$

$d = 12$

β: $3x + 3y + \cancel{2z}\, 2z + 12 = 0$

b) $\underbrace{\begin{pmatrix}1\\-1\\0\end{pmatrix}}_{\sim\vec{AB}} \times \underbrace{\begin{pmatrix}2\\0\\3\end{pmatrix}}_{\sim\vec{AE}} = \begin{pmatrix}-3\\-3\\2\end{pmatrix} = \vec{n}_\alpha$

$\cos\varphi = \frac{\vec{n}_\alpha \cdot \vec{n}_\beta}{n_\alpha \, n_\beta} = \frac{\begin{pmatrix}3\\3\\2\end{pmatrix}\begin{pmatrix}-3\\-3\\2\end{pmatrix}}{\sqrt{22}\,\sqrt{22}}$

$\cos\varphi = \frac{-9-9+4}{22} = -\frac{14}{22}$

$\underline{\varphi = 129{,}5°}$

c) n: $\vec{X} = \underbrace{\begin{pmatrix}7\\5\\-4\end{pmatrix}}_{P} + \lambda\begin{pmatrix}-3\\-3\\2\end{pmatrix}$

$\cos\alpha = \frac{\begin{pmatrix}7\\5\\-4\end{pmatrix}\begin{pmatrix}-3\\-3\\2\end{pmatrix}}{\sqrt{49+25+16}\,\sqrt{9+9+4}} = \frac{-21-15-8}{\sqrt{90}\,\sqrt{22}} = \frac{44}{\sqrt{1980}}$

$\sin\alpha = \sqrt{1-\cos^2\alpha} = \frac{\sqrt{5}}{15}$

$\alpha = 171{,}43°$

$d = \overline{OP} \cdot \sin\alpha = 1{,}41$

$\underline{d = \sqrt{90} \cdot \frac{\sqrt{5}}{15} = \sqrt{2}}$

d) $L(2|2|8)$ Pyr. $(ABEL)$ $\vec{AL} = \begin{pmatrix}6\\2\\8\end{pmatrix}$

$V = \frac{1}{6}\left|\vec{AL}\cdot(\vec{AB}\times\vec{AE})\right| = \frac{1}{6}\left|\vec{AL}\cdot\vec{n}_\alpha\right| = \frac{1}{6}\left|\begin{pmatrix}6\\2\\8\end{pmatrix}\cdot\begin{pmatrix}-24\\-24\\16\end{pmatrix}\right| = \frac{1}{6}\left|-144-48+128\right|$

$\underline{V = \frac{32}{3}}$

e) $\vec{r}_s = \frac{1}{3}\left[\begin{pmatrix}-4\\0\\0\end{pmatrix} + \begin{pmatrix}0\\-4\\0\end{pmatrix} + \begin{pmatrix}0\\0\\6\end{pmatrix}\right] = \frac{1}{3}\begin{pmatrix}-4\\-4\\6\end{pmatrix}$

$\vec{LS} = \begin{pmatrix}-10/3\\-10/3\\-6\end{pmatrix}$ $\vec{LS} \cdot \vec{n}_\alpha = 8 > 0$, also $\sphericalangle(\vec{LS}, \vec{n}_\alpha) < 90°$,
somit kommt der Strahl $\vec{LS}$
von der dem Ursprung zugewandten
Seite

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1. a) Summe

$$\left.\begin{aligned} 0 &= 0+0 \\ 2 &= 1+0\,;\ 0+1\,;\ 1+1 \\ 9 &= 5+4\,;\ 4+5 \end{aligned}\right\}\ 6 \cdot 1/36 \quad \text{Frank gewinnt}$$

$$\left.\begin{aligned} 1 &= 1+0\,;\ 0+1 \\ 3 &= 3+0\,;\ 0+3\,;\ 1+2\,;\ 2+1 \\ 10 &= 5+5 \end{aligned}\right\}\ 7 \cdot 1/36 \quad \text{Robert gewinnt}$$

$\frac{13}{36}$ $\downarrow$

Unterschied: $\frac{23}{36}$

b) $\mu = n \cdot p = 12 \cdot \frac{6}{36} = \frac{72}{36} = 2$

c) $\sigma = \sqrt{npq} = \sqrt{12 \cdot \frac{23}{36} \cdot \frac{13}{36}} = 1{,}664$

d) P(in n Tagen gewinnt R. mind. einmal) > 99%

1 − P(in n Tagen " " gar nicht) > 0,99

$1 - \left(\frac{29}{36}\right)^n > 0{,}99$

$n > \log_{29/36} 0{,}01 = 21{,}3$

$n \geq 22$

e) F R

F:

$$\left.\begin{aligned} (0,0) &: p \cdot 0{,}5 \\ (2,0) &: \tfrac{1-p}{5} \cdot 0{,}5 \\ (0,2) &: p \cdot 0{,}1 \\ (1,1) &: \tfrac{1-p}{5} \cdot 0{,}1 \\ (5,4),\ (4,5) &: 2 \cdot \tfrac{1-p}{5} \cdot 0{,}1 \end{aligned}\right\}$$

$$\frac{p}{2} + \frac{1-p}{10} + \frac{p}{10} + \frac{1-p}{10} + \frac{1-p}{25} =$$

$$= \frac{9}{25}p + \frac{6}{25} \rightarrow \text{max für } \underline{p = 1}$$

wird: $\frac{15}{25} = \frac{3}{5} = 0{,}6$

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5. a) $f(x) = ax^3 + bx^2 + cx + d$ $\qquad f'(x) = 3ax^2 + 2bx + c$

Ursprung: $d = 0$

$$\left.\begin{array}{ll} \text{I} & f(1) = 10 \\ \text{II} & f(2) = 2 \\ \text{III} & f'(1) = 0 \\ \text{IV} & f'(3) = -24 \end{array}\right\} \text{überbestimmt} \qquad \left.\begin{array}{ll} \text{I.} & a + b + c = 10 \\ \text{II.} & 8a + 4b + 2c = 2 \\ \text{III.} & 3a + 2b + c = 0 \\ \text{IV} & 27a + 6b + c = -24 \end{array}\right\} \begin{array}{l} a = 1 \\ b = -12 \\ c = 21 \end{array}$$

Probe ✓

$$\underline{f(x) = x^3 - 12x^2 + 21x}$$

b) $2\sin x = q$: $1 + q + q^2 + q^3 + \ldots = \sum_{i=0}^{\infty} q^i = \frac{1}{1-q}$ geom. Reihe konvergiert für $|q| < 1$

$|q| < 1$

$2|\sin x| < 1$

$|\sin x| < \frac{1}{2} \rightarrow \quad x \in [0; \frac{\pi}{6}[\cup] \frac{5}{6}\pi; \frac{7}{6}\pi[\cup] \frac{11}{6}\pi; 2\pi]$

c) $\lim\limits_{x \to 0} \frac{\sin 3x}{e^{2x} - 1} = \lim\limits_{x \to 0} \frac{3\cos 3x}{2e^{2x}} = \underline{\frac{3}{2}}$

d) $a \sin x = a \cos x$ $\qquad y_1' = a\cos x$ $\qquad y_1'(\frac{\pi}{4}) = a \cdot \frac{1}{2}\sqrt{2}$

$x = \frac{\pi}{4} + z \cdot \pi$ $\qquad y_2' = -a\sin x$ $\qquad y_2(\frac{\pi}{4}) = -a\frac{1}{2}\sqrt{2}$

$$y_1'(\tfrac{\pi}{4}) \cdot y_2'(\tfrac{\pi}{4}) = -1$$
$$-a^2\frac{1}{2} = -1$$
$$\underline{a = \pm\sqrt{2}}$$

e) Induktionsanfang: $n = 1$ $\qquad 3^2 + 4^2 = 25 \mid 5$ ✓

Induktionsannahme: n : $\qquad 3^{2n} + 4^{n+1} = k \cdot 5 \quad ; k \in N$

Induktionschritt : $n+1$: $\qquad 3^{2(n+1)} + 4^{n+1+1}$

$= 3^{2n} \cdot 3^2 + 4^{n+1} \cdot 4$ $\qquad 3^{2n} = 5k - 4^{n+1}$

$= (5k - 4^{n+1}) \cdot 9 + 4^{n+1} \cdot 4$

$= 5 \cdot k \cdot 9 - 9 \cdot 4^{n+1} + 4 \cdot 4^{n+1}$

$= \underline{5}k \cdot 9 - \underline{5} \cdot 4^{n+1} \quad \mid 5$ ✓

Wenn es für n gilt, dann auch für n+1. Es gilt für n=1, also für alle n.